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Clebsch–Gordan coefficients for the permutation group

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Abstract. A non-genealogical method is proposed for the reduction of the inner product representations of the permutation group S_N. This method of determining the Clebsch–Gordan coefficients has been found to be recursive only within a given series. As such it permits the direct reduction of products of large-dimensional representations.

1. Introduction

The role of unitary groups and their irreducible representations (irreps) in the study of many-particle systems has gained considerable importance in recent years (Moshinsky 1968, Paldus 1976, Harter and Patterson 1976, Sarma and Rettrup 1977). The development of useful algorithms for these studies was considerably helped by exploiting the duality between permutation and unitary groups (Robinson 1961, Kaplan 1975). This naturally leads us to attempt a better understanding of the algebraic structure of permutation groups so that the duality may be exploited further.

One of the complex features of the permutation groups is the structure of its inner product representations. A decomposition of the inner products is complicated by the fact that the group is not simply reducible. Secondly, the basis spanning the irreps of the permutation group S_N is realised in a genealogical manner (Hamermesh 1962, Kaplan 1975) and this also adds to the complications. The most detailed studies of the Clebsch–Gordan (CG) coefficients to date relating the basis states of the product representation of S_N to those of the irreducible representations have been due to Hamermesh (1962) and Schindler and Mirman (1977). A programme based on the latter studies was also presented recently. There have also been recent studies of CG coefficients for crystallographic space groups by van den Broek and Cornwell (1978) and Dirl (1979). In addition, a detailed study of the coupling coefficients for finite groups was undertaken recently by Butler (1975). This last article discusses in detail the symmetries of the coupling coefficients without restricting them to form elements of a real orthogonal matrix. Not only the coupling coefficients but also the recoupling coefficients for non-simply reducible finite groups have been dealt with extensively at a formal level. The procedures due to Hamermesh (1962) and Schindler and Mirman (1977) are essentially genealogical in nature and require considerable information on the subgroups of S_N up to S_{N-1}. This implies that if we are interested in the CG coefficients for the reduction of a given inner product of S_N we have to store and retrieve a considerable amount of information. On the other hand, the direct method for reduction of inner products of representations of crystallographic space groups (van den Broek and Cornwell 1978, Dirl 1979) leads to extremely compact expressions for the CG coefficients. These results were obtained using tensor operators defined in terms of triple products of the irreps of the group. The only difficulty in this approach arises from the definition of the tensor operators which require summation over group elements (cf equation (17) of van den Broek and Cornwell 1978 and equations (II.6) and (II.17) of Dirl 1979). For the point group content of the space groups this poses no problems since the order is quite low.

In this paper we consider a scheme for the determination of CG coefficients for S_N which is recursive only within the CG series occurring for the reduction of a given inner product. The procedure does not need any information on S_{N-1} and this in itself means a considerable reduction in storage. The direct methods due to van den Broek and Cornwell (1978) and Dirl (1979) are also difficult to use for S_N in view of the fact that both the order and the dimensionalities of the representation matrices grow rapidly with N. Simplifications were effected in the procedure by noting that we needed only elementary transpositions and orthogonality relations for determining these coefficients. The procedure is outlined in § 2 and illustrated using examples. A brief discussion is presented in § 3.

2. Reduction of inner product representations of S_N

Consider the inner product $[\mu]\times[\nu]$ of two irreps $[\mu]$, $[\nu]$ of S_N of dimensionalities f^{μ}, f^{ν} respectively. This product representation is reducible into irreducible components $[\lambda]$ of S_N as

$$[\mu]\times[\nu]=\sum_{\lambda} a_{\lambda}^{\mu\nu}[\lambda] \tag{1}$$

where $a_{\lambda}^{\mu\nu}$ is the multiplicity of occurrence of $[\lambda]$ in $[\mu]\times[\nu]$. For convenience we may assume that the right side of the above CG series is ordered such that $[\lambda]$ precedes $[\lambda']$ if the first non-zero difference $\lambda_i-\lambda'_i$ $(i=1,2,\ldots,p)$ of the row lengths of the Young shapes (Hamermesh 1962) of $[\lambda]$ and $[\lambda']$ is positive.

Equation (1), in turn, implies that the basis states $|\mu;i\rangle\times|\nu;j\rangle$ $(i=1,\ldots,f^{\mu};j=1,\ldots,f^{\nu})$ are related to $|\lambda\tau_{\lambda};\rho\rangle$ $(\rho=1,\ldots,f^{\lambda})$ through

$$|\lambda\tau_{\lambda};\rho\rangle=\sum_{i,j}\begin{pmatrix}\lambda\tau_{\lambda} & \mu & \nu\\ \rho & i & j\end{pmatrix}|\mu;i\rangle|\nu;j\rangle, \tag{2}$$

where the auxiliary index τ_{λ} has been introduced to distinguish between the multiply occurring $[\lambda]$. The factors $\begin{pmatrix}\lambda\tau_{\lambda} & \mu & \nu\\ \rho & i & j\end{pmatrix}$ on the right of equation (2) are the CG coefficients occurring in this particular reduction.

If P is any element of S_N, we find that equation (2) leads to the result (Hamermesh 1962)

$$\sum_{j,l}[P]_{ij}^{(\mu)}[P]_{kl}^{(\nu)}\begin{pmatrix}\lambda\tau_{\lambda} & \mu & \nu\\ \rho & j & l\end{pmatrix}=\sum_{\rho'}[P]_{\rho'\rho}^{(\lambda)}\begin{pmatrix}\lambda\tau_{\lambda} & \mu & \nu\\ \rho' & i & k\end{pmatrix} \tag{3}$$

where $[P]_{kj}^{(\mu)}$ etc are assumed to be elements of real orthogonal representation matrices of S_N. Noting that equation (3) is linear in the CG coefficients weighted by real elements of the representation matrices, we find that it is possible to choose the CG coefficients to form elements of a real orthogonal matrix (cf Hamermesh 1962 p 260). This leads to

$$\sum_{\lambda,\tau_\lambda,\rho} \begin{pmatrix} \lambda\tau_\lambda & \mu & \nu \\ \rho & i & j \end{pmatrix} \begin{pmatrix} \lambda\tau_\lambda & \mu & \nu \\ \rho & i' & j' \end{pmatrix} = \delta_{ii'}\delta_{jj'} \tag{4}$$

$$\sum_{i,j} \begin{pmatrix} \lambda\tau_\lambda & \mu & \nu \\ \rho & i & j \end{pmatrix} \begin{pmatrix} \lambda'\tau_{\lambda'} & \mu & \nu \\ \rho' & i & j \end{pmatrix} = \delta_{\lambda\lambda'}\delta_{\tau_\lambda\tau_{\lambda'}}\delta_{\rho\rho'}. \tag{5}$$

A direct method for the determination of the CG coefficients of S_N is to use equation (3) for all permutations $P \in S_N$ and the orthogonality relations of equations (4) and (5) (Kaplan 1975). This, however, is a formidable task for all but the lowest values of N since the number and dimensionalities of the required representation matrices increase rapidly with increasing N. It was essentially for this reason that the genealogical methods (Hamermesh 1962, Schindler and Mirman 1977) were developed.

We now propose an alternative scheme. As a first step consider the identity representation of the outer product

$$S_{\lambda_1} \otimes S_{\lambda_2} \otimes \ldots \otimes S_{\lambda_p} \subseteq S_N\Big(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p;\ \sum_i \lambda_i = N\Big)$$

where $[\lambda] \equiv [\lambda_1, \lambda_2, \ldots, \lambda_p]$ is some irrep occurring in the inner product reduction of equation (1). This outer product representation has the dimension (Robinson 1961)

$$f^{[\lambda_1]\otimes[\lambda_2]\otimes\ldots\otimes[\lambda_p]} = \frac{N!}{\lambda_1!\lambda_2!\ldots\lambda_p!}, \tag{6}$$

and can be decomposed into the irreps of S_N as

$$[\lambda_1] \otimes [\lambda_2] \otimes \ldots \otimes [\lambda_p] = \sum_i b_i[\lambda^{(i)}] \tag{7}$$

where b_i is the multiplicity of occurrence of $[\lambda^{(i)}]$ in the outer product. If we order the right side of equation (7) such that $[\lambda] = [\lambda^{(1)}] < [\lambda^{(2)}] < \ldots < [\lambda^{(k)}] = [N]$, then $[\lambda^{(1)}]$ and $[\lambda^{(k)}]$ occurs only once in the reduction (Robinson 1961). Let us consider an irrep $[\lambda]$ occurring in the CG series of equation (1) as induced from the outer product of equation (7). In this process we observe that not only $[\lambda]$ but all irreps higher than it also occur in the outer product reduction. If we consider the highest row symmetry $[\lambda] \equiv [\lambda^{(1)}]$ (say) in the CG series as induced from an outer product of its rows, the only one occurring in this outer product which also occurs in the CG series is this $[\lambda^{(1)}]$. This implies that in order to generate $[\lambda^{(1)}]$ we use only the subspace spanned by the irreps $[\lambda_1^{(1)}] \otimes [\lambda_2^{(1)}] \otimes \ldots \otimes [\lambda_{p_1}^{(1)}]$ of $S_{\lambda_1^{(1)}} \otimes S_{\lambda_2^{(1)}} \otimes \ldots S_{\lambda_{p_1}^{(1)}} \subseteq S_N$. If, further, we assign the first $\lambda_1^{(1)}$ particles to $[\lambda_1^{(1)}]$, the next $\lambda_2^{(1)}$ to $[\lambda_2^{(1)}]$ and so on, we find that the state corresponding to $[\lambda_1^{(1)}] \otimes [\lambda_2^{(1)}] \otimes \ldots \otimes [\lambda_{p_1}^{(1)}]$ has a contribution only from the first of the Young basis (in the ordering of Hamermesh (1962)) spanning $[\lambda^{(1)}]$. The others do not admit such a high row symmetry as this basis state. This means that the first basis state $|[\lambda^{(1)}]\tau_\lambda^{(1)}; 1\rangle$ can be specified (to within an ambiguity due to multiplicity) by applying the permutations of $S_{\lambda_1^{(1)}} \otimes S_{\lambda_2^{(1)}} \otimes \ldots \otimes S_{\lambda_p^{(1)}}$ to $|\mu; i\rangle|\nu; j\rangle$ of equation (1) and assuming the product is invariant under all these permutations. This procedure reduces considerably the effort involved in the determination of the CG coefficients leading to $|[\lambda^{(1)}]\tau_{\lambda^{(1)}}; 1\rangle$.

As an illustration, consider the reduction of the inner product $[2^2, 1] \times [2, 1^3]$ of S_5. From the CG series for this product (Wybourne 1970) we find that $[\lambda^{(1)}] \equiv [4, 1]$ occurs only once in the reduction. Omitting the multiplicity index we represent the first basis of this irrep as

$$|[4,1];1\rangle \equiv \left|\begin{matrix}1234\\5\end{matrix}\right\rangle$$

and the standard tableaux spanning $[2^2, 1]$, $[2, 1^3]$ as

$$[2^2,1]:\ 1\equiv\begin{matrix}12\\34,\\5\end{matrix}\qquad 2\equiv\begin{matrix}12\\35,\\4\end{matrix}\qquad 3\equiv\begin{matrix}13\\24,\\5\end{matrix}\qquad 4\equiv\begin{matrix}13\\25,\\4\end{matrix}\qquad 5=\begin{matrix}14\\25\\3\end{matrix}$$

$$[2,1^3]:\ 1\equiv\begin{matrix}12\\3\\4\\5\end{matrix},\qquad 2\equiv\begin{matrix}13\\2\\4\\5\end{matrix},\qquad 3\equiv\begin{matrix}14\\2\\3\\5\end{matrix},\qquad 4\equiv\begin{matrix}15\\2\\3\\4\end{matrix}.$$

If the permutations of S_5 are to be used for determining the CG coefficients, we need $5! = 120$ of them. However, using the arguments given earlier we observe that we need only $4! = 24$ of them for determining

$$\begin{pmatrix}[4,1] & [2^2,1] & [2,1^3]\\ 1 & i & j\end{pmatrix}.$$

Imposing the requirement that the right-hand side of equation (5) be invariant under each of these permutations and using the corresponding representation matrices of S_5 we obtain the result

$$\begin{pmatrix}[4,1] & [2^2,1] & [2,1^3]\\ 1 & 2 & 1\end{pmatrix}=\begin{pmatrix}[4,1] & [2^2,1] & [2,1^3]\\ 1 & 4 & 2\end{pmatrix}=\begin{pmatrix}[4,1] & [2^2,1] & [2,1^3]\\ 1 & 5 & 3\end{pmatrix},$$

all others being zero. Since we are ultimately left with one unknown, it can be fixed by normalisation leading to

$$|[4,1];1\rangle = 3^{-1/2}(|[2^2,1];2\rangle|[2,1^3];1\rangle + |[2^2,1];4\rangle|[2,1^3];2\rangle + |[2^2,1];5\rangle|[2,1^3];3\rangle).$$

The CG coefficients for $[2^2, 1]\times[2, 1^3]$ leading to other states $|[4, 1]; \rho\rangle (\rho = 2, 3, 4)$ are obtainable by successively applying the elementary transpositions (4, 5), (3, 4), (2, 3) to both sides of the above result and using the transformation properties of the Young basis under these transpositions.

In working with this and more complicated examples we noticed that we needed only the elementary transpositions of $S_{\lambda_1^{(1)}} \otimes S_{\lambda_2^{(1)}} \otimes \ldots \otimes S_{\lambda_{p_1}^{(1)}}$ in order to determine the CG coefficients leading to $|[\lambda^{(1)}]\tau_{\lambda^{(1)}}; 1\rangle$ in any given inner product series. Since a general result of this nature would imply the need for using only $N - p_1$ elementary transpositions of this subgroup of S_N, we were led to establish this result in a general manner as in the Appendix.

The result established in the Appendix and the arguments given above help considerably in determining the CG coefficients of $|[\lambda^{(1)}]\tau_{\lambda^{(1)}}; 1\rangle$ occurring in the reduction of $[\mu]\times[\nu]$. Further, the Young representation matrices have a very simple structure for elementary transpositions (Hamermesh 1962). Using these representation matrices for an elementary transposition $(k,\ k+1)$ $(k=1,\ldots,\lambda_1^{(1)}-1;\lambda_1^{(1)}+1,\ldots,\lambda_1^{(1)}+\lambda_2^{(1)}-1;\ldots)$, which relates the standard tableaux t_i^μ, t_j^ν to $t_{\mathbf{i}}^\mu$, $t_{\mathbf{j}}^\nu$ respectively in equation (3) we readily obtain the results

$$\begin{pmatrix}\lambda^{(1)}\tau_{\lambda^{(1)}} & \mu & \nu \\ 1 & i & \mathbf{j}\end{pmatrix} = \frac{1}{(\tau_i^\mu(k)+\tau_j^\nu(k))}\left\{[1-(\tau_j^\nu(k))^2]^{1/2}\begin{pmatrix}\lambda^{(1)}\tau_{\lambda^{(1)}} & \mu & \nu \\ 1 & i & j\end{pmatrix} - [1-(\tau_i^\mu(k))^2]^{1/2}\begin{pmatrix}\lambda^{(1)}\tau_{\lambda^{(1)}} & \mu & \nu \\ 1 & \mathbf{i} & \mathbf{j}\end{pmatrix}\right\}, \tag{8}$$

$$\begin{pmatrix}\lambda^{(1)}\tau_{\lambda^{(1)}} & \mu & \nu \\ 1 & \mathbf{i} & j\end{pmatrix} = \frac{1}{(\tau_i^\mu(k)+\tau_j^\nu(k))}\left\{[1-(\tau_i^\mu(k))^2]^{1/2}\begin{pmatrix}\lambda^{(1)}\tau_{\lambda^{(1)}} & \mu & \nu \\ 1 & i & j\end{pmatrix} - [1-(\tau_j^\nu(k))^2]^{1/2}\begin{pmatrix}\lambda^{(1)}\tau_{\lambda^{(1)}} & \mu & \nu \\ 1 & \mathbf{i} & \mathbf{j}\end{pmatrix}\right\} \tag{9}$$

where

$$\tau_i^\mu(k)=\pm 1/d_i^\mu(k), \qquad \tau_j^\nu(k)=\pm 1/d_j^\nu(k), \tag{10}$$

with $d_i^\mu(k)$ $(d_j^\nu(k))$ as axial distances between k and $k+1$ in $t_i^\mu(t_j^\nu)$. On the right-hand sides of equation (10) the $-(+)$ sign is to be used if k is an entry in a row preceding (succeeding or same row as) $k+1$ in the given tableau.

The procedure carried through above for the CG coefficients leading to $|[\lambda^{(1)}],\tau_{\lambda^{(1)}};1\rangle$ could equally well have been applied to the last irrep $[\lambda^{(f)}]$ in the CG series of equation (1). If $|[\lambda^{(f)}]\tau_{\lambda^{(f)}};l\rangle$ is the last of the Young–Yamanouchi basis for $[\lambda^{(f)}]$ which has minimum row symmetry in the CG series, we note that there is no other tableau of any other irrep which has greater column antisymmetry. Let $\bar\lambda_1^{(f)}, \bar\lambda_2^{(f)}, \ldots \bar\lambda_p^f$ be the column lengths of the Young shape corresponding to this irrep. Then, as before, $|[\lambda^{(f)}]\tau_{\lambda^{(f)}};l\rangle$ can be determined (to within the ambiguity due to multiplicity) using the transpositions of $[1^{\bar\lambda_1^{(f)}}]\otimes[1^{\bar\lambda_2^{(f)}}]\otimes\ldots\otimes[1^{\bar\lambda_{pf}^{(f)}}]$ where the first $\bar\lambda_1^{(f)}$ particles define the first $[1^{\bar\lambda_1^{(f)}}]$ and so on. In this case we have $(k,k+1)|[\lambda^{(f)}]\tau_{\lambda^{(f)}};l\rangle=(-1)|[\lambda^{(f)}]\tau_{\lambda^{(f)}};l\rangle$ for $k=1,\ldots,\bar\lambda_1^{(f)}-1;\ \bar\lambda_1^{(f)}+1,\ldots,\bar\lambda_1^{(f)}+\bar\lambda_2^{(f)}-1;\ldots$. For this case equations (8) and (9) become

$$\begin{pmatrix}\lambda^{(f)}\tau_{\lambda^{(f)}} & \mu & \nu \\ l & i & \mathbf{j}\end{pmatrix} = \frac{1}{\tau_j^\nu(k)-\tau_i^\mu(k)}\left([1-(\tau_j^\nu(k))^2]^{1/2}\begin{pmatrix}\lambda^{(f)}\tau_{\lambda^{(f)}} & \mu & \nu \\ l & i & j\end{pmatrix} + [1-(\tau_i^\mu(k))^2]^{1/2}\begin{pmatrix}\lambda^{(f)}\tau_{\lambda^{(f)}} & \mu & \nu \\ l & \mathbf{i} & \mathbf{j}\end{pmatrix}\right) \tag{11}$$

$$\begin{pmatrix} \lambda^{(f)}\tau_{\lambda^{(f)}} & \mu & \nu \\ l & i & j \end{pmatrix} = \frac{1}{\tau_i^{\mu}(k)-\tau_j^{\nu}(k)}\left([1-(\tau_i^{\mu}(k))^2]^{1/2}\begin{pmatrix} \lambda^{(f)}\tau_{\lambda^{(f)}} & \mu & \nu \\ l & i & j \end{pmatrix} + [1-(\tau_j^{\nu}(k))^2]^{1/2}\begin{pmatrix} \lambda^{(f)}\tau_{\lambda^{(f)}} & \mu & \nu \\ l & i & j \end{pmatrix}\right) \tag{12}$$

where τ_j^{μ} etc are as defined in equation (10).

As an illustration, using $[2^2, 1]\times[2, 1^3]$ we can readily obtain $\left|\begin{smallmatrix} 1 & 4 \\ 2 & 5 \\ 3 & \end{smallmatrix}\right\rangle$ as

$$\left|\begin{matrix} 1 & 4 \\ 2 & 5 \\ 3 & \end{matrix}\right\rangle = \frac{1}{2\sqrt{22}}(|[2^2, 1]; 1\rangle|[2, 1^3]; 2\rangle + \sqrt{3}|[2^2, 1]; 2\rangle|[2, 1^3]; 2\rangle - |[2^2, 1]; 3\rangle|[2, 1^3]; 1\rangle - \sqrt{3}|[2^2, 1]; 4\rangle|[2, 1^3]; 1\rangle)$$

where the tableau indexing is the same as the one used in the previous example. The procedure for determining the CG coefficients leading to other tableaux of this irrep follows as before.

The fact that we can start at either end of the CG series reduces the effort of calculating the CG coefficients for any irrep in a given series considerably. We now consider this problem.

Let us consider an irrep $[\lambda^{(k)}]$ $(k \neq 1, f)$ which occurs in the CG series in the reduction of a $[\mu]\times[\nu]$. Without loss of generality we may assume that it is closer to $[\lambda^{(1)}]$ than $[\lambda^{(f)}]$. For the basis state $|[\lambda^{(k)}]; 1\rangle$ of this irrep, we can, using the procedure developed earlier, apply the $N-p_k$ elementary transpositions belonging to $S_{\lambda_1^{(k)}} \otimes S_{\lambda_2^{(k)}} \otimes \ldots \otimes S_{\lambda_{p_k}^{(k)}}$. But now we note from equation (7) that a reduction of the outer product of the identity irrep of this subgroup leads to the occurrence of $[\lambda^{(k-1)}]$, $[\lambda^{(k-2)}], \ldots, [\lambda^{(1)}]$ with various multiplicities. These irreps occur also in the CG series. Hence the state obtained using the elementary transpositions of this subgroup has contributions in it from all these irreps. If we have determined the CG coefficients leading to the basis for all these higher irreps, we can eliminate these contributions by successive orthogonalisations. At the final stage we will be left with an essential set of unknowns equal in number to the multiplicity of occurrence of $[\lambda^{(k)}]$ in $[\mu]\times[\nu]$. The linear combinations which occur with the final set of unknown coefficients are linearly independent by construction. A choice in this ambiguous situation (Hamermesh 1962) is to choose each of these groups as defining a linearly independent basis state spanning $[\lambda^{(k)}]$ occurring in $[\mu]\times[\nu]$. Schmidt orthogonalisation of these basis states leads to the orthonormal set $|[\lambda^{(k)}]\tau_{\lambda^{(k)}}; 1\rangle$ which is the required one. The rest of the basis states $|[\lambda^{(k)}]\tau_{\lambda^{(k)}}; \rho\rangle$ $(\rho \neq 1)$ can then be obtained as in the case of $|[\lambda^{(1)}]\tau_{\lambda^{(1)}}; \rho\rangle$. If $[\lambda^{(k)}]$ is closer to $[\lambda^{(f)}]$ we can go through the procedure starting from $|[\lambda^{(f)}]\tau_{\lambda^{(f)}}; l\rangle$. In this sense the procedure is recursive within a CG series defined by a given $[\mu]\times[\nu]$ of S_N.

As an illustration of the complications which can arise in all the above procedures consider the reduction of $[5, 2, 1^2]\times[4, 3, 1^2]$ of S_9 which leads to the CG series (Wybourne 1970)

$$[5, 2, 1^2]\times[4, 3, 1^2] = [8, 1] + 3[7, 2] + 3[7, 1^2] + \ldots + [2, 1^7].$$

The determination of the basis state $|[8,1];1\rangle$ follows readily as in the previous example on using the $N-p_1=7$ transpositions $(1,2),(2,3),\ldots(7,8)$ in equations (5), (8) and (9). Using the procedures outlined earlier we obtain the result

$$|[8,1];1\rangle\equiv\left|\begin{matrix}12345678\\9\end{matrix}\right\rangle$$

$$=\frac{1}{3\sqrt{10}}\sum_{i\in[4,2,1^2]}|[5,2,1^2];(i1)\rangle|[4,3,1^2];(i2)\rangle \tag{13}$$

where $(i1)$ designates one of the 90 standard tableaux of the S_8 substructure $[4,2,1^2]$ contained in $[5,2,1^2]$ with 9 in the first row. Similarly $(i2)$ refers to the same subtableaux structure as for $[5,2,1^2]$ occurring in $[4,3,1^2]$ with 9 in the second row. The basis states for the other 7 standard tableaux spanning [8, 1] can be obtained by applying (8, 9) to both sides of equation (13), generating $\left|\begin{smallmatrix}12345679\\8\end{smallmatrix}\right\rangle$ and applying (7, 8) to this state and so on. As an illustration, applying (8, 9) to both sides of equation (15) we obtain

$$-\frac{1}{8}\left|\begin{matrix}12345678\\9\end{matrix}\right\rangle+\frac{\sqrt{63}}{8}\left|\begin{matrix}12345679\\8\end{matrix}\right\rangle$$

$$=(8,9)\Big(\sum_{i'\in[3,2,1^2]}|[5,2,1^2];(i'11)\rangle|[4,3,1^2];(i'12)\rangle$$

$$+\sum_{i'\in[4,1^3]}|[5,2,1^2];(i'21)\rangle|[4,3,1^2];(i'22)\rangle$$

$$+\sum_{i'\in[4,2,1]}|[5,2,1^2];(i'41)\rangle|[4,3,1^2];(i'42)\rangle\Big)$$

$$=\sum_{i'\in[3,2,1^2]}|[5,2,1^2];(i'11)\rangle\Big(-\frac{1}{2}|[4,3,1^2];(i'12)\rangle$$

$$+\frac{\sqrt{3}}{2}|[4,3,1^2];(i'21)\rangle\Big)+\sum_{i'\in[4,1^3]}\Big(\frac{1}{4}|[5,2,1^2];(i'21)\rangle$$

$$+\frac{\sqrt{15}}{4}|[5,2,1^2];(i'12)\rangle\Big)\times|[4,3,1^2];(i'22)\rangle$$

$$+\sum_{i'\in[4,2,1]}\Big(\frac{1}{7}|[5,2,1^2];(i'41)\rangle+\frac{\sqrt{48}}{7}|[5,2,1^2];(i'14)\rangle\Big)$$

$$\times\Big(\frac{1}{4}|[4,3,1^2];(i'42)\rangle+\frac{\sqrt{15}}{4}|[4,3,1^2];(i'24)\rangle\Big) \tag{14}$$

where i' is the common subtableau structure of the irreps of S_7 indicated on the summation sign and 21, 12 etc indicate the row occurrences of the entries 8 and 9 in the respective irreps of S_9. Using the result from equation (13) for the first term on the left of equation (14) and rearranging and simplifying we obtain

$$\left|\begin{matrix}12345679\\8\end{matrix}\right\rangle = \frac{1}{9\sqrt{70}}\Big[\sum_{i'\in[3,2,1]}(-3\,|[5,2,1^2];(i'11)\rangle|[4,3,1^2];(i'12)\rangle$$
$$+4\sqrt{3}|[5,2,1^2];(i'11)\rangle|[4,3,1^2];(i'21)\rangle)$$
$$+\sum_{i'\in[4,1^3]}(3|[5,2,1^2];(i'21)\rangle|[4,3,1^2];(i'22)\rangle$$
$$+2\sqrt{15}|[5,2,1^2];(i'12)\rangle|[4,3,1^2];(i'22)\rangle)$$
$$+\frac{1}{7}\sum_{i'\in[4,2,1]}(9|[5,2,1^2];(i'41)\rangle|[4,3,1^2];(i'42)\rangle$$
$$+8\sqrt{3}|[5,2,1^2];(i'14)\rangle|[4,3,1^2];(i'42)\rangle$$
$$+2\sqrt{15}|[5,2,1^2];(i'41)\rangle|[4,3,1^2];(i'24)\rangle$$
$$+24\sqrt{5}|[5,2,1^2];(i'14)\rangle|[4,3,1^2];(i'24)\rangle)\Big]. \quad (15)$$

Consider now the irrep [7, 2] occurring in the CG series for $[5,2,1^2]\times[4,3,1^2]$. The first of the standard Young tableaux $\begin{smallmatrix}1234567\\89\end{smallmatrix}$ is maximally invariant under the permutations of the subgroup $S_7\otimes S_2$. Choosing a linear combination of products $|[5,2,1^2];i\rangle|[4,3,1^2];j\rangle$ which is invariant under the transpositions (1, 2), (2, 3), . . . , (6, 7), (8, 9) of this subgroup we obtain

$$\psi = \sum_{i\in[3,2,1^2]}(i11|i12)|[5,2,1^2];(i11)\rangle\Big(\frac{1}{2}|[4,3,1^2];(i12)\rangle+\frac{\sqrt{3}}{2}|[4,3,1^2];(i21)\rangle\Big)$$
$$+\sum_{i\in[4,1^3]}(i12|i22)\Big(\frac{\sqrt{3}}{2\sqrt{2}}|[5,2,1^2];(i12)\rangle$$
$$+\frac{\sqrt{5}}{2\sqrt{2}}|[5,2,1^2];(i21)\rangle\Big)\times|[4,3,1^2];(i22)\rangle$$
$$+\sum_{i\in[4,2,1]}(i14|i24)\Big(\frac{\sqrt{3}}{\sqrt{7}}|[5,2,1^2];(i14)\rangle$$
$$+\frac{2}{\sqrt{7}}|[5,2,1^2];(i41)\rangle\Big)\times\Big(\frac{\sqrt{3}}{2\sqrt{2}}|[4,3,1^2];(i24)\rangle$$
$$+\frac{\sqrt{5}}{2\sqrt{2}}|[5,2,1^2];(i42)\rangle\Big)+\sum_{i\in[4,2,1^2]}(i14|i24)$$
$$\times\Big(\frac{2}{\sqrt{7}}|[5,2,1^2];(i14)\rangle-\frac{\sqrt{3}}{\sqrt{7}}|[5,2,1^2];(i41)\rangle\Big)$$
$$\times\Big(\frac{\sqrt{5}}{2\sqrt{2}}|[4,3,1^2];(i24)\rangle-\frac{\sqrt{3}}{2\sqrt{2}}|[4,3,1^2];(i42)\rangle\Big) \quad (16)$$

where the notation used is the same as for the previous examples except for the coefficients $(i\alpha\beta|i\gamma\delta)$. The products of these with the respective numerical factors in the summations can be identified with the CG coefficients leading to $\left|{}^{1234567}_{89}\right\rangle$ of [7, 2] if the overlaps with the states of higher symmetry [8, 1] irrep are eliminated. We find that we need only to orthogonalise $|[8,1];1\rangle$ of equation (13). This process leaves us with three unknowns which cannot be unambiguously determined. The linear combination of $|[5,2,1^2];i\rangle|[4,3,1^2];j\rangle$ which occurs with each of the unknowns is linearly independent of the others. This allows us to use the Schmidt procedure to obtain the orthonormal basis states listed below:

$$\left|\begin{matrix}1234567\\89\end{matrix}\right\rangle(1)=\frac{1}{2}\left(\frac{7}{85}\right)^{1/2}\Bigg[\frac{1}{2\sqrt{2}}\sum_{i\in[4,1^3]}(\sqrt{3}|[5,2,1^2];(i12)\rangle+\sqrt{5}|[5,2,1^2];(i21)\rangle)$$
$$\times|[4,3,1^2];(i22)\rangle-\frac{\sqrt{10}}{7}\sum_{i\in[3,2,1^2]}|[5,2,1^2];(i11)\rangle$$
$$\times(|[4,3,1^2];(i12)\rangle+\sqrt{3}|[4,3,1^2];(i21)\rangle)\Bigg]$$

$$\left|\begin{matrix}1234567\\89\end{matrix}\right\rangle(2)=\frac{1}{3}\left(\frac{17}{105}\right)^{1/2}\Bigg[\frac{-35}{34\sqrt{14}}\sum_{i\in[4,1^3]}(\sqrt{3}|[5,2,1^2];(i12)\rangle+\sqrt{5}|[5,2,1^2];(i21)\rangle)$$
$$\times|[4,3,1^2];(i22)\rangle-\frac{1}{17}\left(\frac{35}{2}\right)^{1/2}\sum_{i\in[3,2,1^2]}|[5,2,1^2];(i11)\rangle$$
$$\times(|[4,3,1^2];(i12)\rangle+\sqrt{3}|[4,3,1^2];(i21)\rangle)$$
$$+\frac{1}{2\sqrt{14}}\sum_{i\in[4,2,1]}(\sqrt{3}|[5,2,1^2];(i14)\rangle+2|[5,2,1^2];(i41)\rangle)$$
$$\times(\sqrt{3}|[4,3,1^2];(i24)\rangle+\sqrt{5}|[4,3,1^2];(i42)\rangle)\Bigg]$$

$$\left|\begin{matrix}1234567\\89\end{matrix}\right\rangle(3)=\frac{\sqrt{3}}{14\sqrt{35}}\Bigg[-\frac{7\sqrt{5}}{18}\sum_{i\in[4,1^3]}(\sqrt{3}|[5,2,1^2];(i12)\rangle+\sqrt{5}|[5,2,1^2];(i21)\rangle)$$
$$\times|[4,3,1^2];(i22)\rangle-\frac{7}{9}\sum_{i\in[3,2,1^2]}|[5,2,1^2];(i11)\rangle$$
$$\times(|[4,3,1^2];(i12)\rangle+\sqrt{3}|[4,3,1^2];(i21)\rangle)$$
$$+\sum_{i\in[4,2,1]}\Bigg(\frac{5\sqrt{5}}{3}|[5,2,1^2];(i14)\rangle|[4,3,1^2];(i24)\rangle$$
$$-\frac{23\sqrt{3}}{9}|[5,2,1^2];(i14)\rangle|[4,3,1^2];(i42)\rangle$$
$$-\frac{11\sqrt{15}}{9}|[5,2,1^2];(i41)\rangle|[4,3,1^2];(i24)\rangle$$
$$+\frac{17}{9}|[5,2,1^2];(i41)\rangle|[4,3,1^2];(i42)\rangle\Bigg)\Bigg].$$

The other basis states for this irrep can be derived starting from the above using the same procedure as for $|[8,1];\rho\rangle$.

As a final illustration we consider the state

$$\left| \begin{array}{ll} 1 & 9 \\ 2 & \\ 3 & \\ 4 & \\ 5 & \\ 6 & \\ 7 & \\ 8 & \end{array} \right\rangle$$

of $[2, 1^7]$. This state is totally antisymmetric under the permutations of $S_8 \otimes S_1$. Using the transpositions (1, 2), (2, 3), . . . , (7, 8) in equations (5), (11) and (12) we can readily obtain the result

$$\left| \begin{array}{ll} 1 & 9 \\ 2 & \\ 3 & \\ 4 & \\ 5 & \\ 6 & \\ 7 & \\ 8 & \end{array} \right\rangle = \frac{1}{3\sqrt{10}} \sum_{i \in [4,2,1^2]} \Lambda_i^{[4,2,1^2]} |[5, 2, 1^2]; (i1)\rangle |[4, 3, 1^2]; (\tilde{i}2)\rangle$$

where $\Lambda_i^{[4,2,1^2]}$ is $+1$ or -1 depending on whether i is obtained from the first tableau of the set using an even or odd permutation and $\tilde{i}$ is the conjugate tableau to i. The procedure for obtaining the states of $[2^2, 1^5]$ etc is the same as before.

3. Discussion

The combined use of outer and inner product reductions used in § 2 has led to a reasonably straightforward method for determining the CG coefficients for S_N. There is no doubt that the procedure is not as simple as for the simply reducible groups. This has been brought out by Hamermesh (1962), Schindler and Mirman (1977) and Butler (1975). To the best of our knowledge the procedure developed here is the most direct method for determining these coefficients for S_N.

The fairly complicated example of the reduction of $[5, 2, 1^2] \times [4, 3, 1^2]$ which was worked out in some detail was chosen to bring out some of the salient features of the method. Firstly, such an example could not have been attempted using earlier methods of Hamermesh (1962) or Schindler and Mirman (1977) without a considerable amount of stored information on the subgroups of S_9. Also this example could not be handled using the other direct methods (van den Broek and Cornwell 1978, Dirl 1979) since the order of S_9 is quite large. Secondly, the fact that only elementary transpositions and orthogonality relations are needed for the determination of the CG coefficients is of considerable computational advantage. One more advantage of the procedure is that at every stage we have to solve only linear equations. Finally, the fact that a given irrep of the CG series can be reached from either end means that the computational effort is reduced by half. It has to be admitted, however, that the ambiguity arising from multiplicity has been resolved in a particular way (Hamermesh 1962) which may not always be the most suitable one. In this context it is to be noted that the procedure due to Dirl (1979) also does not lead to unambiguous identification for multiply occurring irreps for the group S_N. Finally, we have not been able to use the complex conjugation and permutational symmetries of the CG coefficients (Butler 1975) for two reasons. Firstly, a choice of real coefficients (possible in the cases of S_N) obviates the need for coupling conjugation symmetries. Secondly, ours is a technique for attempting a reduction of a specific product $[\mu]\times[\nu]$ leading to a given CG series. This implies that we can omit consideration of permutation symmetries except when considering tabulation of these quantities.

We are attempting to develop a computer program based on the present work.

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Appendix

Consider a functional defined over the permutations $P\in S_N$ as

$$\phi_{ik\rho}^{\mu\nu\lambda\tau_\lambda}(P)=\sum_{j,l}[P]_{ij}^{(\mu)}[P_k]_l^{(\nu)}K_{\rho jl}^{\lambda\tau_\lambda\mu\nu}-\sum_{\rho'}[P]_{\rho'\rho}^{(\lambda)}K_{\rho' ik}^{\lambda\tau_\lambda\mu\nu} \tag{A1}$$

where $[P]^{(\mu)}$, $[P]^{(\nu)}$, $[P]^{(\lambda)}$ are the real orthogonal Young representation matrices for a $P\in S_N$ and $K_{\rho ik}^{\lambda\tau_\lambda\mu\nu}$ are (as yet undefined) unknowns independent of $P\in S_N$. Let us assume that a set of values have been assigned to these unknowns leading to a fixed numerical value for $\phi_{ik\rho}^{\mu\nu\lambda\tau_\lambda}(R)$ where R is an elementary transposition $(\alpha,\alpha+1)(\alpha=1,\ldots,N-1)$ of S_N. Let R, S be two such elementary transpositions and

$$P=RS. \tag{A2}$$

We now observe that $\phi_{ik\rho}^{\mu\nu\lambda\tau_\lambda}(P)$ can be expressed as

$$\begin{aligned}\phi_{ik\rho}^{\mu\nu\lambda\tau_\lambda}(P)&=\phi_{ik\rho}^{\mu\nu\lambda\tau_\lambda}(RS)\\&=\sum_{j,l}\sum_{j'l'}[R]_{ij}^{(\mu)}[R]_{kl}^{(\nu)}K_{\rho jl}^{\lambda\tau_\lambda\mu\nu}[S]_{j'j}^{(\mu)}[S]_{l'l}^{(\nu)}-\sum_{\rho'}\sum_{\rho''}[R]_{\rho'\rho''}^{(\lambda)}[S]_{\rho''\rho}^{(\lambda)}K_{\rho' ik}^{\lambda\tau_\lambda\mu\nu}\\&=\sum_{j'l'}[R]_{ij'}^{(\mu)}[R]_{kl'}^{(\nu)}\Big(\phi_{j'l'\rho}^{\mu\nu\lambda\tau_\lambda}(S)+\sum_{\rho'}[S]_{\rho'\rho}^{(\lambda)}K_{\rho' j'l'}^{\lambda\tau_\lambda\mu\nu}\Big)-\sum_{\rho'\rho''}[R]_{\rho'\rho''}^{(\lambda)}[S]_{\rho''\rho}^{(\lambda)}K_{\rho' ik}^{\lambda\tau_\lambda\mu\nu}\\&=\sum_{j,l}[R]_{ij}^{(\mu)}[R]_{kl}^{(\nu)}\phi_{jl\rho}^{\mu\nu\lambda\tau_\lambda}(S)+\sum_{\rho'}[S]_{\rho'\rho}^{(\lambda)}\phi_{ik\rho'}^{\mu\nu\lambda\tau_\lambda}(R).\end{aligned} \tag{A3}$$

Since the right-hand side of equation (A3), by assumption, is completely determined by just the elementary transpositions we find that there exist no new values of $K_{\rho ik}^{\lambda\tau_\lambda\mu\nu}$ which are consistent with these values of $\phi_{ik\rho}^{\mu\nu\lambda\tau_\lambda}(RS)$, $\phi_{jk\rho}^{\mu\nu\lambda\tau_\lambda}(S)$ and $\phi_{ik\rho'}^{\mu\nu\lambda\tau_\lambda}(R)$. This argument can be extended to any $P\in S_N$ since all of them can be expressed as products of elementary transpositions. That the converse is also true may be readily verified. Any assigned values for the function defined in terms of elementary transpositions R, S lead to a set of relations among the unknowns. If P is any permutation expressible as in equation (A2), we observe that there is only one possible value for $\phi_{ik\rho}^{\mu\nu\lambda\tau_\lambda}(RS)$ which follows from the right-hand side of equation (A3). Thus $P = RS$ does not define any new relation among the unknowns.

This result being generally true for any values of the functionals, it is also true when the unknowns on the right-hand side of equation (A1) are identified with CG coefficients as in equation (5) leading to an assigned value zero for the functionals for all $P \in S_N$. Thus independent relations among the CG coefficients are possible only with elementary transpositions of S_N.

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